WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540, August, 2016

Prove all results. You may quote standard named results proved in class.

1. Suppose (X, Y) represents the coordinates of a point selected at random from the square $[-1/2, 1/2] \times [-1/2, 1/2]$. Find the pdf of the random variable $Z = XY$.

2. A fair coin is flipped. Suppose the coin is flipped until HT appears in two successive flips. Let X denote the total number of flips for this to happen. Give a formula for the probability mass function of X.

3. Let $X \sim \text{Normal}(0, 1)$. Let $Z \sim \text{Bernoulli}(p)$, $f(y \mid X, Z = 1) = \delta_x(y)$, the indicator function, and $f(y \mid X, Z = 0) = N_y(0, 1)$. Show that X and Y are identically distributed. Find the correlation between X and Y.

4. Let $f(y) = 2\Phi(\lambda y)\phi(y)$, where λ is any real number, $\phi(\cdot)$ is the standard normal density, and $\Phi(\cdot)$ its cumulative density function. Find the probability density function of Y^2. Are there surprises? Explain.

5. Let $X_1,\ X_2$ be two random variables. Let the probability density function (pdf) of X_2 be given by

$$f_2(X_2) = c_2 X_2^4, \quad 0 < x_2 < 1.$$

Let the conditional pdf of X_1 given $X_2 = x_2$ be given by

$$f_{1|2}(x_1|x_2) := \begin{cases} c_1 x_1 / x_2^2, & 0 < x_1 < x_2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c_1 and c_2.

(b) Calculate $\text{Var}(X_1)$, $\text{Var}(E(X_1|X_2))$ and $E(\text{Var}(X_1|X_2))$, and verify

$$\text{Var}(X_1) = \text{Var}((E(X_1|X_2)) + E(\text{Var}(X_1|X_2)).$$